Proceedings of the 31st Conference
on Decision and Control
Tucson, Arizona • December 1992

A CONSTRUCTIVE SOLUTION TO THE GAIN–PHASE MARGIN PROBLEM *

Juan C. Cockburn
Department of Electrical Engr.
University of Minnesota
Minneapolis, MN 55455

Yariv Sidar
Department of Electrical Engr.
Israel Institute of Technology
Haifa, Israel 32000

Allen Tannenbaum
Department of Electrical Engr.
University of Minnesota and
Israel Institute of Technology

Abstract

In this paper we present a constructive solution to the General Robust Stabilization Problem presented in [4]. For that we use Nevanlinna–Pick interpolation theory and numerical conformal mappings. Then we define and solve the gain-phase margin optimization problem as a special case of the General Problem. Finally we give some examples to illustrate our procedure.

1 Notation

$\mathbf{C}$	complex numbers
$\mathbf{H}$	open right half plane = $\{s \in \mathbf{C} : Re\ s > 0\}$
$\overline{\mathbf{H}}$	closed right half plane = $\{s \in \mathbf{C} : Re\ s \geq 0\}$
$\tilde{\mathbf{H}}$	extended right half plane = $\overline{\mathbf{H}} \cup \{\infty\}$
$\mathbf{D}$	open unit disk = $\{s \in \mathbf{C} : \lvert s\rvert < 1\}$
$\overline{\mathbf{D}}$	closed unit disk = $\{s \in \mathbf{C} : \lvert s\rvert \leq 1\}$

2 Introduction

The maximum obtainable stability margins give a measure of how difficult a feedback system is to control. In particular, the gain margin and phase margin are two classical measures of robustness against independent gain or phase perturbations, respectively. Here we will study combined gain and phase perturbations. First, we will give an algorithm to solve the General Problem proposed in [4]. For that we will use Nevanlinna–Pick (N-P) interpolation theory and numerical conformal mappings. Then we will solve the gain-phase margin problem and define an optimal gain-phase margin as a measure of robustness. Finally we present some examples. All the results are for single-input single-output (SISO) finite dimensional linear time-invariant (FDLTI) systems.

3 Robust Stabilization Problems

In general terms the problem we want to solve can be stated as follows: Given $P_k(s)$, a parameterized family of plants, where k is a parameter vector taking values in some compact set $\mathbf{K}$, find (if possible) a proper controller $C(s)$ such that for every $k \in \mathbf{K}$ the closed loop of Fig. 1. is internally asymptotically stable. Here $\mathbf{K}$ can be regarded as a representation of the modeling uncertainty. In many practical problems of parametric uncertainty P_k can be modeled as $P_k(s) = kP_o(s)$ where $P_o(s)$ is a fixed nominal plant and $k \in \mathbf{K} \subset \mathbf{C}$ compact, $\mathbf{K}$ simply connected. Some practical problems that fall in this category are:

1. The gain margin problem
$$\mathbf{K} = \{k \in [a, b], b > 1 > a > 0\}.$$
2. The phase margin problem
$$\mathbf{K} = \{k = e^{j\phi}, \phi \in [\phi_1, \phi_2], 0 \leq \phi_i \leq \pi, i = 1, 2\}.$$
3. The gain-phase margin problem
$$\mathbf{K} = \{k = re^{j\phi}, r \in [a, b], \phi \in [\phi_1, \phi_2] \text{ as above}\}.$$

The following lemma from [4] gives a precise statement of our robust stabilization problem along with necessary and sufficient conditions for a solution.

Lemma 1 *Let $P_k(s) = kP_o(s)$. $P_o(s)$ has right half plane (RHP) zeros $\{z_1, \ldots, z_m\}$ (∞ possibly included) and RHP poles $\{p_1, \ldots, p_n\}$ and $k \in \mathbf{K} \subset \mathbf{C}$ compact, simply connected. Assume further that $0 \notin \mathbf{K}$ and $1 \in \mathbf{K}$. Let $S(s) = [1 + P_o(s)C(s)]^{-1}$ be the usual sensitivity function. Then a stabilizing controller $C(s)$ exists for all $P_k(s)$ if and only if*

i) $S(s)$ is analytic in $\tilde{\mathbf{H}}$ and satisfies the interpolation conditions

$$\begin{cases} S(z_i) = 1, & i = 1, \ldots, m \\ S(p_j) = 0, & j = 1, \ldots, n \end{cases} \tag{1}$$

ii) $S(s)$:$\tilde{\mathbf{H}} \to \mathbf{G} := \mathbf{C} \setminus \{\frac{k}{k-1} : k \in \mathbf{K}\}$

This robust stabilization problem is a particular case of the following abstract problem first proposed in [4].

*This work was supported in parts by grants from the National Science Foundation DMS-8811084, the Air Force Office of Scientific Research AFOSR-90-0024, and the Army Research Office DAAL03-91-G-0019.

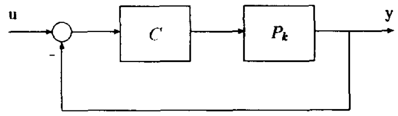

Figure 1: The Standard Unity Feedback System

General Problem Let $\mathbf{G} \subset \mathbf{C}$ be a given simply connected domain with $\{0, 1\} \in \mathbf{G}$. Find (if possible) a rational analytic function $S(s)$:$\tilde{\mathbf{H}} \to \mathbf{G}$ satisfying the interpolation conditions (1).

Remark: Note that the robust stabilization problem is a particular case of this General Problem, simply choose $\mathbf{G} := \mathbf{C} \setminus \{\frac{k}{k-1} : k \in \mathbf{K}\}$.

$$\begin{tikzcd} \tilde{\mathbf{H}} \arrow[r, "S"] \arrow[d, "\psi"'] & \mathbf{G} \arrow[d, "\theta"] \\ \mathbf{D} \arrow[r, "\tilde{S}"'] & \mathbf{D} \end{tikzcd}$$

Figure 2: Commutative Diagram

The General Problem can be solved by mapping it to an equivalent Nevanlinna–Pick interpolation problem as the commutative diagram of Fig. 2 illustrates. The mapping $\psi(s) = \frac{s-1}{s+1}$ is fixed, and θ is a conformal mapping we must construct. For the above problem to have a solution the following must be true:

Theorem 1 *The General Problem is solvable if and only if $|\theta(1)| < \alpha_{max}$ where*

$$\alpha_{max}^{-1} = \inf_C \|S\|_\infty$$

The two key steps in solving the General Problem are: 1) the computation of the invariant α_{max}, which depends only on the unstable zeros and poles of P_o and 2) the computation of $|\theta(1)|$, which depends on the uncertainty $\mathbf{G}$ and requires the construction of a conformal equivalence $\theta : \mathbf{G} \to \mathbf{D}$. The following procedure solves the General Problem:

Input: $P_o(s), \mathbf{K}$
Step 1. From the interpolation data compute α_{max}.
Step 2. Find $\mathbf{G}$ and construct the conformal equivalence θ.
Step 3. If $|\theta(1)| < \alpha_{max}$ stop, problem has no solution.
Step 4. Solve N-P problem to obtain $\tilde{S}(s)$.
Step 5. Compute the frequency response $\tilde{S}(j\omega)$.
Step 6. Using θ^{-1} compute $S(j\omega)$.
Step 7. Find a real rational approximant of $S(s)$.

The computation of α_{max} is a classic problem in N-P interpolation theory [1, 4]. Our main concern is the construction of the conformal equivalence θ. This is described in the next section.

4 Construction of the Conformal Equivalence

While the Riemann Mapping Theorem tells us that the conformal equivalence θ exists, the proofs are not constructive. A conformal equivalence between a simply connected region and the disk is called a *Riemann Mapping*. There are two classical methods to construct numerical Riemann Mappings: 1) *Osculation methods* and 2) *Schwarz-Christoffel transformations*. An algorithm by Caratheodory, that belongs to the first class, was used first by [7] and later by [6] to solve the gain-phase margin problem. However, osculation methods converge very slowly [3]. On the other hand, Schwarz-Christoffel transformations involve the solution of a nonlinear system of equations and complex integrals which make them delicate numerical problems [9].

The new algorithm that we present here was recently developed by Marshall and Morrow [5]. It is based on a sequence of simple conformal maps

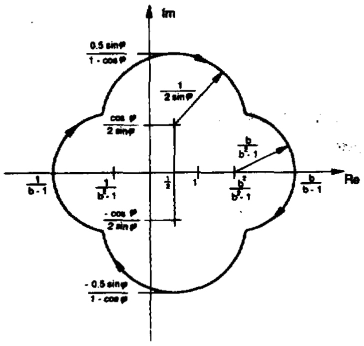

Figure 3: Region G for the Gain-Phase Margin Problem

allowing us to obtain the conformal map θ in one pass. The Marshal and Morrow algorithm works efficiently for arbitrary regions G bounded by Jordan Curves.

With the construction of the conformal mapping θ out of the way we proceed to discuss the gain-phase margin problem, an important practical measure of robustness.

5 The Gain-Phase Margin Problem

In general, the gain-phase margin problem arises when the perturbation set $\mathbf{K}$ is given by

$$\mathbf{K} = \{\ k = re^{j\phi}, \quad r \in [a,b], b > 1 > a > 0: \quad \phi \in [\phi_1, \phi_2], 0 \leq \phi_i \leq \pi, i = 1,2\} \tag{2}$$

We can define the gain margin (GM) and phase margin (PM) as

$$GM = 20\log_{10}(\sqrt{b/a}) \tag{3}$$

$$PM = \frac{180^\circ}{\pi}\frac{[\phi_2 + \phi_1]}{2} \tag{4}$$

By setting $a = 1/b$ and $\phi = \phi_2 = \phi_1$ we obtain the classic gain-phase margin problem. In this case, the set G is as shown in Fig. 3.

The gain-phase margin problem can be stated as follows: Given a desired gain and phase margin pair (b, ϕ) and a nominal plant $P_o(s)$, find a proper controller $C(s)$ such that $kP_o(s)$ remains stable for all perturbations k in $\mathbf{K}$. If this is so the closed loop system will have a combined gain-phase margin of $20\log_{10}(b)$ dB and $\frac{180}{\pi}\phi$ degrees.

Note that the gain margin problem and phase margin problem are limiting cases of the gain-phase margin problem:

as $\phi \to 0, \mathbf{G} \to \mathbf{C} \setminus (-\infty, \frac{1}{1-b}] \cup [\frac{b}{1-b}, \infty)$

as $b \to 1, \mathbf{G} \to \mathbf{C} \setminus (-\infty, \frac{1}{2} - j\frac{1}{2}\frac{\sin\phi}{1-\cos\phi}] \cup [\frac{1}{2} + j\frac{1}{2}\frac{\sin\phi}{1-\cos\phi}, \infty)$

Moreover, it is easy to show [6] that

$$|\theta(1)| = \begin{cases} \frac{b-1}{b+1} & ,\phi = 0; \\ \sin(\phi/2) & ,b = 1. \end{cases} \tag{5}$$

Now, we can define the following gain-phase margin optimization problem: Find a pair $(b, \phi)_{opt}$ that maximizes in some sense the gain-phase margin. Before defining what an optimal gain-phase margin is let us look at some definitions of optimality [1]:

Lemma 2 *The optimal gain margin b_{opt} is given by*

$$b_{opt} = \begin{cases} \infty & , P_o \text{ stable or minimum phase;} \\ \frac{1+\alpha_{max}}{1-\alpha_{max}} & , \text{else.} \end{cases}$$

Lemma 3 *The optimal phase margin ϕ_{opt} is given by*

$$\phi_{opt} = \begin{cases} \pi & , P_o \text{ stable or minimum phase;} \\ 2\sin^{-1}(\alpha_{max}) & , \text{else.} \end{cases}$$

Note that when $|\theta(1)| = \alpha_{max}$ we attain the optimal gain margin b_{opt} and phase margin ϕ_{opt}, respectively. However in the gain-phase margin problem the condition $|\theta(1)| = \alpha_{max}$ does not define a unique optimal solution since $(b_{opt}, 0)$ and $(1, \phi_{opt})$ both satisfy the above condition. To solve this ambiguity we proposed the following optimal gain-phase margin:

Definition 1 *Let $\beta := b_{opt}/\phi_{opt}$, with b_{opt} and ϕ_{opt} as above. Then the optimal gain-phase margin is the pair $(b, \phi)_{opt}$ such that $b/\phi = \beta$ and $|\theta(1)| = \alpha_{max}$*

The rationale behind the above definition is to maintain the best possible gain and phase margins achievable simultaneously. Many other practical measures of optimality can be defined.

Now we are ready to show how to solve the gain-phase margin problem using the algorithm in section 4. We plan to give a longer discussion of the algorithm of Marshall-Morrow since we believe it may be useful for many practical engineering applications.

6 Examples

Example 1 Let $P_o(s) = \frac{10(s-5)}{(s-1)(s-3)}$. Find the optimal gain-phase margin.

For simplicity we will not consider the zeros at ∞ as interpolation points. In this case this will not affect the value of α_{max}. The interpolation data is a = { 5, 1, 3}, b ={1, 0, 0}. We first compute $\alpha_{max} = 0.1667$ and also $\beta = 1.4/0.3349 = 4.180$. Now $(b, \phi)_{opt}$ can be obtained by a simple binary search. At each iteration we construct the mapping θ and compute $|\theta(1)|$ until $||\theta(1)| - \alpha_{max}| < \epsilon$, ϵ a desired tolerance. For $\epsilon = 0.001$ and after 9 iterations $(b, \phi)_{opt} = (1.049, 0.251) = (0.42\text{dB}, 14.4\text{deg})$.

Example 2 Given $P_o(s) = \frac{10(s-5)}{(s+2)(s-1)(s-3)}$. Find a strictly proper stabilizing controller $C(s)$ such that the closed loop has a gain-phase margin of (0.73dB,4.8 deg).

The interpolation data is the same as in the previous example, so $\alpha_{max} = 1.667$. For the given gain-phase margin $|\theta(1)| = 1.5$. Now we construct the N-P solution to obtain $\widetilde{S}(s)$. Then we compute $\widetilde{S}(j\omega)$ and using θ^{-1} map it back to $S(j\omega)$. The last step is to find a reduced order real rational approximation to $S(j\omega)$. For this we use the GKL algorithm [2]. The parameters chosen where $\lambda = 3$,M=1024 (IFFT points), N=100 and R=20. Finally $C(s)$ was obtained from $P_o(s)[S^{-1}(s) - 1]$ and cascading the filter $J(s) = \frac{5000^3}{(s+5000)^3}$ to make $C(s)$ strictly proper. Fig. 4 shows a Nyquist plot of the optimal loop gain function $L(s) = P_o(s)C(s)$.

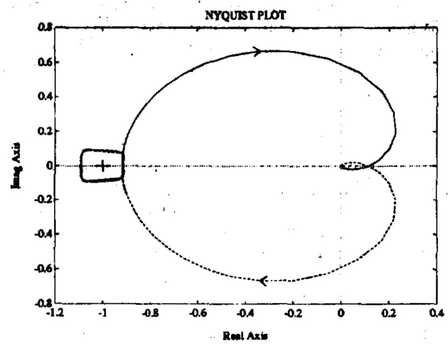

Figure 4: Region G or the Gain-Phase Margin Problem

7 Conclusions

We have presented an algorithm to solve certain class of robust stabilization problems where the uncertainty is independent of frequency. This amounts to solving a N-P interpolation problem together with the construction of a conformal mapping, the main difficulty being the construction of this mapping. Our contribution has been to bring to the control community a simple algorithm to construct approximate Riemann Mappings. Finally we presented the gain-phase margin problem and solved it using the proposed algorithm. We are currently working in the generalization of the above results to multivariable robust stabilization problems.

References

[1] Doyle J., B. Francis and A. Tannenbaum, *Feedback Control Theory*, MacMillan 1991.

[2] Gu G., P. Khargonekar and E.B. Lee, "Approximation of Infinite-dimensional Systems", *IEEE Trans Autom. Contr.*, AC-34, 6, June 1989.

[3] Henrici P., *Applied and Computational Complex Analysis III.* Wiley, NY, 1986.

[4] Khargonekar P. and A. Tannenbaum, "Non-Euclidean Metrics and the Robust Stabilization of Systems with Parameter Uncertainty", *IEEE Trans Autom. Contr.*, AC-30, 10, Oct. 1985.

[5] Marshal D. and J. Morrow, Personal communication. 1987.

[6] Sidar Y., "The Combined Problem of Gain and Phase Margins in Linear Control Systems - Theoretical Boundaries Using Nevanlinna-Pick Interpolation Theory", MS Thesis, Technion, Haifa - Israel, May 1990.

[7] Sideris A. and G. Safonov, "A Design Algorithm for the Robust Synthesis of SISO Feedback Control Systems Using Conformal Maps and H_∞ Theory", *Proc. ACC*, pp. 1710-1715, Sept. 1984.

[8] Tannenbaum A., *Invariance and System Theory: Algebraic and Geometric Aspects*, New York: Springer-Verlag, 1981.

[9] Trefethen L.N., "Numerical Computation of the Schwarz-Christoffel Transformation", *SIAM J. Sci. Stat. Comput.*, 1, 1, March 1980.